

* Stoke's Theorem:

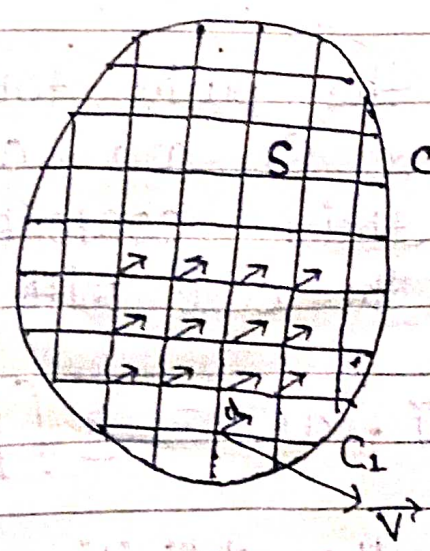
This theorem states that the line integral of a vector \vec{v} around any closed curve C is equal to the surface integral of $\text{curl } \vec{v}$ taken over any surface S of which the curve C is a boundary edge. i.e.,

$$\oint_C \vec{v} \cdot d\vec{l} = \iint_S \text{curl } \vec{v} \cdot d\vec{S} = \iint_S (\nabla \times \vec{v}) \cdot d\vec{S}$$

Proof:

Let a vector \vec{v} be a function of position. Then its line integral along a closed curve C is given by

$$L = \oint_C \vec{v} \cdot d\vec{l}$$



Where $d\vec{l}$ is a small element of path as shown in Fig. Let us divide the enclosed area by drawing lines over it into a large number of small areas dS_1, dS_2, \dots, dS_n bounded by the curves $C_1, C_2, C_3, \dots, C_n$.

Let us suppose that the curve C_1 is divided into two parts to form two separate closed curves C_1' and C_1'' . If L_1 and L_2 are the line integrals for the two parts of the curve under consideration, then

$$L = L_1 + L_2$$

$$\text{or } \oint_C \vec{v} \cdot d\vec{l} = \oint_{C_1'} \vec{v} \cdot d\vec{l} + \oint_{C_1''} \vec{v} \cdot d\vec{l}$$

Since the paths traversed by the two of the curve C_1 are oppositely traced and hence their contribution is cancelled. Similar is the case with other curves.

$$\therefore \oint_C \vec{v} \cdot d\vec{l} = \sum \oint_{C_n} \vec{v} \cdot d\vec{l}$$

But from the definition of curl \vec{v} , we have

$$\oint_C \vec{v} \cdot d\vec{l} = \text{Curl } \vec{v} \cdot d\vec{S}_m$$

Hence the surface area is $d\vec{S}_m$, so we have

$$\text{Curl } \vec{v} \cdot d\vec{S}_m = \oint_{C_m} \vec{v} \cdot d\vec{l}$$

$$\therefore \oint \vec{v} \cdot d\vec{l} = \sum \oint_{C_m} \vec{v} \cdot d\vec{l} = \sum \text{Curl } \vec{v} \cdot d\vec{S}_m$$

The summation on the R.H.S. is clearly the surface integral of $\text{Curl } \vec{v}$ over the entire surface S enclosed by the curve so that we have,

$$\begin{aligned} \oint_C \vec{v} \cdot d\vec{l} &= \iint \text{Curl } \vec{v} \cdot d\vec{S} \\ &= \iint_S (\nabla \times \vec{v}) \cdot d\vec{S} \end{aligned}$$